Ariel University Center of Samaria

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Problems of the Internet Math Olympiad

March 19, 2008

-Before you are 10 problems of various degrees of difficulty, try to solve as many problems as possible.
-Use Word or write your solutions in a clear handwriting.
-Explain your solutions. Solutions without explanations will not be accepted.
-Send your solutions to the e-mail address found on the website.
-You may submit solutions only once.
-The criteria for ranking the solutions are: exactness of solutions and in the case of a tie the amount of time taken to submit the solutions.
-The grading system is based on the principle that the number of points each problem is worth is inversely proportional to the number of contestants who solved this problem.

Problem 1.

Calculate the following limits:

a) $\lim\limits_{n\to\infty}\sum\limits_{k=n}^{2n}\frac{1}{k!}$, b) $\lim\limits_{n\to\infty}\sum\limits_{k=n}^{2n}\frac{1}{k}$.

Problem 2.

Prove that for any 9 interior points of a cube whose sides equal 1, at least two of them can be chosen such that the distance between them does not exceed $\frac{\sqrt{3}}{2}$.

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Problem 3.

a) Calculate the sum of the infinite series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$,

b) Find the following limit: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_1^n \ln\left(1+\frac{1}{\sqrt{x}}\right) dx$

Problem 4.

Prove that $\int_0^{\sqrt{2\pi}} \sin(x^2)\, dx > 0$.

Problem 5.

Prove that for any polynomial $p(x)$ of the degree $n > 1$ and any point Q the number of tangents to the graph of $p(x)$ which pass through the point Q does not exceed n.

Problem 6.

Let A, B, C, D be four distinct spheres in a space. Suppose the spheres A and B intersect along a circle which belongs to some plane P, the spheres B and C intersect along a circle which belongs to some plane Q, the spheres C and D intersect along a circle which belongs to some plane S, and the spheres D and A intersect along a circle which belongs to some plane T. Prove that the planes P,Q,S,T are either parallel to the same line or have a common point.

Problem 7.

For a square matrix A denote $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \frac{A^9}{9!} - \ldots$

a) Prove, that if the matrix A is symmetric $A = A^T$, then all elements of the matrix $\sin A$ belongs to the segment [-1,1].

b) Is the above assertion true for non-symmetric matrix A?

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Problem 8.

All the position of a cellular tape are numerated by the numbers 0,1,2,3,… and in some of them one or more game pieces can be placed. Our moves are determined by the following rules:

1) If in all of the positions whose numbers are $n \geq 1$ there is no more than 1 game piece in each, we add 2 game pieces into position number 1.

2) Otherwise, the position $n \geq 1$ with the maximal number from all the positions which have at least two game pieces is chosen, and then 2 game pieces are moved from this position in two opposite directions: one of them is moved from the chosen position n to the position $n-k$ and another game piece is moved from the chosen position n to the position $n+k$, where k is an integer number $(1 \leq k \leq n)$. This number k can be chosen arbitrary for each move.

What is the maximal number of moves that can be made so that no game piece will be in the positions with numbers greater than 2008?

Problem 9.

A matrix 2008×2008 is given. All its elements equal 0 or 1. Assume that every two lines differ from each other in a half of the positions. Prove that every two columns in this case also differ in a half of the positions.

Problem 10.

Let $\frac{\alpha_n}{\beta_n}$ be an irreducible fraction of the form $\frac{\alpha_n}{\beta_n} = \sum_{k=1}^{n} \frac{1}{k}$.

Let us call the prime number p a good number if it is a divisor of α_n for a some n. Prove that the set of all good numbers p is infinite.